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MATHEMATICAL MODELLING FOR BUCKLING OF BEAM-COLUMN AND ITS ANALYTICAL SOLUTION

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ABSTRACT

In the mathematical modelling, the Differential Equation that governs the Buckling behaviour of Beam-Column is derived and the analytical solution is obtained by using the method of Variation of Parameters. An important application of this model is its usefulness in the analysis of buckling of drillstrings within curved boreholes.

Keywords*: Buckling, Beam-column, Method of Variation of Parametres.*

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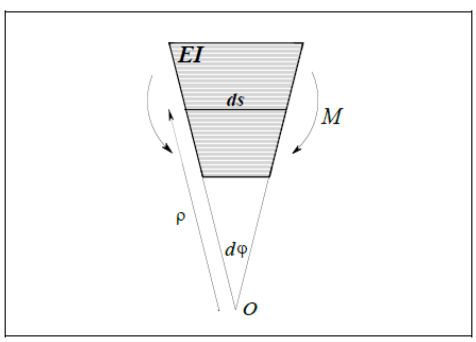
1. INTRODUCTION

We are interested in Buckling behaviour of an beam column and the differential equation governing it. Buckling is the sudden large deformation of structure due to a slight increase of an existing load under which the structure had exhibited little, if any, deformation before the load was increased. Beam-column is the name given to a beam subjected to an axial compressive force in addition to a lateral force (or load).

1.1 Modelling of Beam-Column Equation

The most of the mathematical models for beam and columns are based on Euler-Bernoulli beam theory, which assumes that plane cross sections perpendicular to the axis of the beam before the displacement, remain plane and perpendicular to the axis after deformation.



Under this assumption, and also assuming that the curvature at any point of the centre line depends only on the magnitude of the bending moment at that point, the angle $d\varphi$ between two faces of an infinitesimal element of length ds and stiffness EI, subjected to a bending moment M, is given by

$$\frac{d\emptyset}{ds} = \frac{M}{EI} = \frac{1}{\rho}$$

Where, ρ is the radius of curvature of the centre line of the element. The angle φ relates to $y(x)$ by

$$\emptyset = \tan^{-1}\left(\frac{dy}{dx}\right)$$

If s is the arc length measured along the column, one obtains

$$\frac{dx}{ds} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

Substituting the above equations results immediately in the following nonlinear differential equation:

$$\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{M}{EI}$$

For small displacements and consequently small slopes, the nonlinear differential moment equation may be approximated by the linear differential equation

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

This is the basic differential equation for bending.

Where, EI is cross section stiffness, x is position along the inclined beam-column, y is lateral displacement function.

When a beam-column is subjected to an axial compressive force in addition to a lateral force.

The differential equation is given by

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = q(x) \neq 0$$

$$y(0) = y(l) = y(0)=y(l) = 0.$$

Where, P is the axial force and $q(x)$ the distributed weight. Now the trivial function $y(x) = 0$ is not a solution of the differential equation. We will find the solution for $q(x) = q_0(q_0$ is constant).

2. ANALYTIC SOLUTION OF BUCKLING OF AN BEAM-COLUMN

We apply the method of Variation of Parameters to differential equation of Beam-column to obtain the Analytic Solution.

Solution of ordinary differential equation is given by,

$$y = y_h + y_p$$

Where, y_h is General Solution & y_p is Particular Solution.

General solution is given by,

$$y_h = c_1 + c_2 x + c_3 \cos\sqrt{\frac{P}{EI}}x + c_4 \sin\sqrt{\frac{P}{EI}}x$$

And Particular Solution is given by,

$$y_p = \sum_{k=1}^{n}(x)\int \frac{W_k(x)}{W(x)} r(x)dx$$

Where, W is Wronskian, W_k is obtained from W by replacing the k^{th} column of W by the column [0 0 … 0 1]^T and $r(x) = \frac{q_0}{EI}$

Calculating all the Wronskian we get the particular solution,

$$y_p(x) = y_1 \int \frac{W_1(x)}{W(x)} \frac{q_0}{EI}\, dx + y_2 \int \frac{W_2(x)}{W(x)} \frac{q_0}{EI}\, dx + y_3 \int \frac{W_3(x)}{W(x)} \frac{q_0}{EI}\, dx + y_4 \int \frac{W_4(x)}{W(x)} \frac{q_0}{EI}\, dx$$

$$= \frac{q_0}{P}\left(\frac{x^2}{2} - 1\right)$$

The solution is

$$y = y_h + y_p$$

$$\therefore y(x) = c_1 + c_2 x + c_3 \cos\sqrt{\frac{P}{EI}}x + c_4 \sin\sqrt{\frac{P}{EI}}x + \frac{q_0}{P}\left(\frac{x^2}{2} - 1\right)$$

*N*ow applying first boundary conditions:

$$y(0) = 0 \text{ and } y(l) = 0$$

We have,

$$y'(x) = c_2 - c_3\sqrt{\frac{P}{EI}}\sin\sqrt{\frac{P}{EI}}x + c_4\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}x + \frac{q_0}{P}x$$

$$y''(x) = -c_3 \frac{P}{EI}\cos\sqrt{\frac{P}{EI}}x - c_4 \frac{P}{EI}\sin\sqrt{\frac{P}{EI}}x + \frac{q_0}{P}$$

Now applying second boundary conditions:

$$y''(0) = 0 \text{ and } y''(l)=0$$

We have,

$$y(x) = \frac{q_0}{P} - \frac{q_0 EI}{P^2}\frac{q_0}{2P}(lx) - \frac{q_0}{P} + \frac{q_0 EI}{P^2} cos\sqrt{\frac{P}{EI}}x + \frac{q_0 EI}{p^2}\left(\frac{1-cos\sqrt{\frac{P}{EI}}l}{sin\sqrt{\frac{P}{EI}}l}\right) sin\sqrt{\frac{P}{EI}}x + \frac{q_0}{P}(\frac{x^2}{2})$$

$$= -\frac{q_0}{P}x(l-x) + \frac{q_0 EI}{P^2}\left(\frac{cos\sqrt{\frac{P}{EI}}x sin\sqrt{\frac{P}{EI}}l + sin\sqrt{\frac{P}{EI}}x - cos\sqrt{\frac{P}{EI}}l sin\sqrt{\frac{P}{EI}}x}{sin\sqrt{\frac{P}{EI}}l}\right)$$

The General Solution is,

$$y(x) = \frac{q_0 l^4}{16EIu^4}\left[\frac{cos(u - 2ux/l)}{cos(u)} - 1\right] - \frac{q_0 l^2}{8EIu^2}x(l - x),$$

Where, $u = \frac{l}{2}\sqrt{\frac{P}{EI}}$

This is the Analytical Solution for Buckling of Beam-column.

No solution exists for $u= (k+\frac{1}{2})\pi$ for k a natural number. Therefore, the value $u = \frac{\pi}{2}$ represents the first critical value of the system. From the definition of u we find

$$\frac{\pi}{2} = \frac{l}{2}\sqrt{\frac{P_{crit}}{EI}} \rightarrow P_{crit} = \frac{\pi^2 EI}{l^2}$$

Although this expression is the same as the critical force for an Euler's column, the behavior is significantly different. For the Euler's column it represents an eigen value of the problem and for the beam-column it represents a singularity.

3. CONCLUSION

General column buckling formula for evaluating critical load with, out of plane loading is always govern by boundary conditions of columns. General behaviour of beam-column under loading highlights that equilibrium solution for axial forces less than critical gives stable solution while axial forces more than critical gives always unstable solution. Obtained analytical solution fully agrees with expected behaviour of beam-column.

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