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E- INVERSIVE SEMIGROUPS WITH THE IDENTITY abc = ac

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ABSTRACT

A semigroup S is called an E- inversive if for every $a \in S$ there exists x in S such that $ax \in E(S)$ (i.e., $(ax)^2 = ax$) , where E(S) is the set of all idempotents of S, introduced by G.Thierrin. In this paper some preliminaries and basic concept of E – inversive semigroups are presented and proved that a semigroup S is an E – inversive semigroupif and only if it is an inverse semigroup and an E-inversive semigroup is commutative band if and only if it is both left and right regular . Again it is proved that an E- inversive semigroup with the identity abc = ac for any a ,b ,c in S is one of the following : normal , left (right) regular , left (right) semi normal , left(right) quasi normal . It is also proved that a semilattice is an E - inversive semigroup if and only ifit satisfies the property aba =ab (aba = ba) or ab = a (ab = b).

Keywords: Semigruop, E -inversive ,regular

INTRODUCTION:

The concept of an E – inversive semigroup was constructed by G.Thierrin [1] and developed by Lallement and Petrich [2] and Petrich[3]. But Petrich was studied in some what different form of Lallement and Petrich. This type of semigroups recently reappeared in papers Hall and Munn [4] ,F.Catino and M.M.Miccoli [5],Margolis and Pin [6] and H.Mitsch [7].The special case of an E-inversive semigroups with commuting idempotents called E-dense was considered by Margolis and Pin .We note that the defination is not one- sided.The work presented in this paper is based on the results of idempotent semigroups of F. catino amd M.M .Miccoli[8], G.Shobhalatha [9] ,Rajeswari A[10,11,12]

Preliminaries:

1.1.1 Definition: An element 'a ' of a semigroup S is called an E-inversive if there is an element x in S such that $(ax)^2 = ax$ ie., $ax \in E(S)$. Where E(S) is set of all idempotent elements of S.

1.1.2 Definition: A semigroup S is called an E –inversive semigroup if every element of S is an E-inversive.

Examples:

1. Regular semigroups ($a = axa$ implies that $ax \in E(S)$)
2. Eventually regular semigroups(a^n is regular for some n >1 implies that $a^n x a^n = a^n$ => $a^n x \in E(s)$ for some $x \in S$)

1.1.3 Definition: An E-inversive semigroup is said to be an E-dense if all elements of an E-inversive semigroup are commute.

1.1.4 Definition: A subset A of S is said to be right unitary if for any $a \in A$, $s \in S$ implies $(sa)^2 = sa \in A$, and $s \in A$

1.1.5 Definition : A subset A of S is said to be left unitary if for any $a \in A$, $s \in S$ implies that $(as)^2 = as \in A$, and $s \in A$.

1.1.6 Definition : A subset A of S is said to be unitary if it is both left and right unitary.

1.1.7 Remark: Now we have established the equivalence between the two identities aba = a and abc = ac, on idempotent semigroups thus either one of them define rectangularity. (The structure of idempotent semigroups by Naoki Kimura[8])

1.1.8 Lemma : An element a of a semigroup S is an E-inversive iff there exists $y \in S$ such that y = yay

1.1.9 Lemma : Let S be an idempotent commutative semigroup then S is an E-inversive semigroup.
Proof: Let S be an idempotent commutative semigroup. To prove that S is an E-inversive semigroup we have to prove that every element $a \in S$ is an E-inversive. To show that a is an E-inversive , consider $(ax)^2 = ax.ax = a(xa)x = a(ax)x = (aa)(xx) = (a)(x) = ax$ therfore $(ax)^2 = ax$ for some x in S.Every element a in S is an E –inversive.Hence S is an E -inversive semigroup.

1.1.10Theorem: A Semigroup is an E-inversive semigroup iff it is inverse semigroup
Proof: Let S be a semigroup.Assume that S is an inverse semigroup. Then for any $a \in S$ there exists $a^1 \in S$ such that $aa^1a = a$ and $a^1aa^1 = a^1$.To show that S is an E - inversive semigroup.ie; every element in S is an E - inversive **.** Let a be in S then $(aa^1)^2 = aa^1\, aa^1 = (aa^1\, a)a^1 = aa^1$. $(aa^1)^2 = aa^1$ => a is an E-inversive . Therfore S is an E -inversive semigroup. Conversely, let S is an E-inversive semigroup. Then every element of S is an E-inversive for a in S, there exists an element x in S such that $(ax)^2 = ax$.Let $a \in S$ and $a^1 \in S$. Put $a = xa^1\, x$ for any $x \in S$. Consider $aa^1a = (xa^1x)a^1\ (xa^1x) = (xa^1)\,(xa^1)\,(xa^1)\,x = (xa^1)(xa^1)^2x = (xa^1)(xa^1)x = (xa^1)^2x = (xa^1)x = xa^1x = a$. Therefore $aa^1a = a$ .Similarily we can prove that $a^1aa^1 = a$.Hence S is an inverse semigroup.

1.1.11 Theorem: An E-inversive semigroup is commutative band (Semilattice) iff it is both left and right regular.
Proof: Let S be an E - inversive semigroup**.** Assume that S is commutative band (Semilattice) . Now we show that S is left regular and right regular.Let $a,b \in S \Rightarrow a^2 = a$ and $b^2 = b$. Since S is E- inversive we have $ab = (ab)^2 = ab.ab = ab(ab) = ab(ba) = a(bb)a$
$= ab^2a = aba$ => ab = aba. There fore S is left regular . Again $ab = (ab)^2 = (ab)(ab) = (ba)(ab) = b(aa)b = ba^2b = bab$. Therefore ab = bab => S is right regular .Hence S is both left and right regular band. Conversely, assume that S is both left and right regular .Let $a,b \in S \Rightarrow ab = (ab)^2 = ab.ab = a(bab) = a(ba) = aba = ba$ (Since S is left (right) regular) . Therefore S is Commutative. Now we prove that S is a band .We have $a, x \in S \Rightarrow (ab)^2 = ab$ Put b=a in the above $(a.a)^2 = a.a \Rightarrow (a.a)^2 = a^2 \Rightarrow a.a = a$. ∴ S is band. Hence S is Semilattice.

1.1.12 Note: In the above theorem S is an E-inversive semigroup and is commutative so S is an E-dense semigroup.
1.1.13 Theorem: Let S be a semigroup. If S is left(right) singular then S is an E-inversive semigroup.
Proof: Let S be a semigroup with singular property i.e.,ab = a for any $a,b \in S$.To prove that S is an E-inversive semigroup, we have to prove that every element of S is an E-inversive.

Consider $(ab)^2 = ab.ab = a(ba)b = a.b.b = a(bb) = ab \Rightarrow (ab)^2 = ab$. That is for any a in S there exists an element b in S such that $(ab)^2 = ab \Rightarrow ab \in E(S)$.Therefore ‘a’ is an E-inversive in S .Hence S is an E-inversive semigroup

1.1.14Lemma: An idempotent semigroup S with an identity abc = ac, for any a,b,c∈S is an E-inversive semigroup.

Proof: Let S be an idempotent semigroup with an identity abc = ac where a,b,c∈S abc = ac => abcb = acb. Put c=a then abab = a.ab=>$(ab)^2 = a^2b$ =>$(ab)^2$ = ab .Therefore every element of S is an E-inversive semigroup. Hence S is an E-inversive semigroup

1.1.15Lemma: An E-inversive semigroup S with an identity abc=ac for any a,b,c∈S, is normal.

Proof: Let S be an E-inversive semigroup with an identity abc=ac for all a,b,c∈S. Now we have to show that S is normal. Consider abca = a(bc)a= $a(bc)^2a$ = abcbca= abc(bca) = abcba = (abc)ba = acba => abca = acba. There fore S is normal.

1.1.16 Lemma: An E-inversive semigroup S with an identity abc = ac for any a,b,c∈S is regular.

Proof: Let S be an E-inversive semigroup with an identity abc = ac for all a,b,c∈S. We have to prove that S is regular. Let a,b,c∈S , then abca = ab(ca) = $ab(ca)^2$ = abcaca = a(bca)ca = a(ba)ca =>abca = abaca.Therefore S is regular.

1.1.17 Lemma :An E-inversive semigroup S with an identity abc = ac for any a,b,c∈S is right(left) Semi-regular.

Proof: Let S be an E-inversive semigroup with an identity abc=ac. Now we show that S is right semi-regular . Let a , b, c in S, then abca = a(bc)a = abcbca = ab(cb)ca = abcabca = abca(bc)a = abcabaca =>abca = abcabaca .Hence S is right semi-regular.

1.1.18 Note: Similarily we prove that an E-inversive semigroup S with an identity abc = ac for any a,b,c∈S then S is 1. left(right) Quasi –normal , 2. left(right) semi-normal 3. left (right) normal.

1.2 PROPERTIES OF E –INVERSIVE SEMIGROUPS

This section consists of some theorems on E – inversive semigroups satisfying the identity ab =a

1.2.1 Theorem:A semilattice S is an E-inversive semigroup if and only if it satisfy the identity aba =,ab for any a,b∈S

Proof: Let S be a Semilattice. Assume that S satisfies the identity aba = ab for all a,b∈S. Let aba = ab =>a(ba) = ab =>a(bab) = ab =>(ab)(ab) = ab => $(ab)^2$ = ab i.e., for every a in S there exists an element b in S such that $(ab)^2$ = ab => a is an E – inversive element . Since 'a' is arbitrary element follows that S is an E-inversive semigroup.Conversely, let S be an E-inversive semigroup . Let a in S , then there exists b in S such that ab = $(ab)^2$ = (ab)(ab) = (ab)(ba) = a(bb)a = $a(b)^2a$ = aba =>aba = ab Therefore S satisfies the identity aba = ab for all a,b∈S.

1.2.2 Theorem:A Semilattice S is an E-inversive semigroup if and only if it satisfies the identity ab = a (ba = a) for any a,b∈S.

Proof: Let S be a semilattice **.**Assume that S satisfies the identity ab = a (ba = a) for any a,b∈S. Let ab = a =>(a)b = (a) => abb = ab **=>** a(b)b = ab => abab = ab =>(ab)(ab) = ab=>$(ab)^2$ = ab. That is, for every a in S there exists an element b in S such that $(ab)^2$ = ab=>a is an E - inversive. Every element of S is an E-inversive .Hence S is an E-inversive semigroup. Conversely, let S be a semilattice .Assume that S is an E –inversive semigroup then for any a , b in S ab = $(ab)^2$ => ab = (ab)(ab) = (ab)(ba) = a(bb)a = ab^2a = aba = a => ab = a. Hence S satisfies the identity ab = a for any a,b∈S.

1.2.3 Theorem : A Commutative total semigroup S is left (right) regular if and only if S is an E-inversive semigroup

Proof: Let S be a Commutative total semigroup **.** Assume that S is left regular then aba = ab =>a(ba) = ab =>a(bab) = ab =>(ab)(ab) = ab =>$(ab)^2$ = ab i.e., for every a in S there exists an element b in S such that $(ab)^2$ = ab => a is an E-inversive. Hence S is an E-inversive semigroup. Conversely, let S is E-inversive semigroup . Since S is total every element of S can be written as product of two elements in S i.e., S^2 = S. Let x in S then x=ab for some a ,b in S . Since S is E – inversive semigroup and x in S implies by 1.1.8 there exists c in S such that xcx = x=> (ab)c(ab) = ab =>(ab)c(ba) = ab =>a(bcb)a = ab => a(b)a = ab => aba = ab. Therefor S is left regular.

1.2.4 Note: For any total semigroup S which is commutative semigroup is an E-inversive semigroup if and only if S is left(right) singular.

1.2.5 Theorem: An E-inversive semigroup S is regular ifand only if it is left(right) semi-regular

Proof: Let S be an E-inversive semigroup. Assume that S is regular then abca = abaca for any a,b,c$\in$S .Let abca = aba(ca) = $aba(ca)^2$ = abacaca = abac(a)ca = abacabaca = abac(abaca) = abacabca =>abca = abacabca .Therefore S is left semi-regular. Conversely, let S be a left semi-regular, then abca = acbca = a(cbc)a =aca = (a)ca = abaca => abca = abaca . Hence S is regular

1.2.6 Theorem: Let S be an E-inversive semigroup then S is left(right) semi -regular

Proof: Let S be an E-inversive semigroup, then for any a , b, c in S and since S is regular => abca = abaca = (aba)ca= $aba(ca)^2$ =abacaca = abac(a)ca = abacabaca = abac(abaca) = abacabca=> abca = abacabca .Therefore S is left semi regular .

1.2.7 Lemma: An E-inversive semigroup S is left(right) semi-normal if and only if it is left(right) quasi-normal.

Proof: let S be an E-inversive semigroup S.Assume that S is left semi-normal then abca = acbca for any a,b,c$\in$S .Let abca = acbca for all a,b,c$\in$S .abcac =acbcac =>ab(cac) = acb(cac) =>abc = acbc . Therfore S is left quasi normal . Conversely let S be a left quasi-normal then abc = acbc =>abca =acbca. Therefore S is left semi-normal.

1.2.8 Lemma: Let S be an E-inversive semigroup and E(s) is set of all idempotent elements of S then E(S) is unitary.

Proof: Let S be an E-inversive semigroup and E(S) is set of all idempotent elements of S.Now we show that E(S) is unitary. Let a be an element of E(s) then there exist an element b in S such that $(ab)^2$= ab => ab$\in$E(s).Now we show that, b$\in$E(S) => $(a.b)^2$ = a.b , put a = b then $(b.b)^2$ = b.b =>$(b.b)^2$ = b^2 => b.b = b =>b^2 = b. Therefore b$\in$E(S). Hence E(S) is left unitary. Similarly we can prove that, E(s) is right unitary. Therefore E(s) is unitary.

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