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CONVOLUTION FOR LAPLACE-WEIERSTRASS TRANSFORM

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ABSTRACT

Mathematics plays a vital role in the expanding knowledge of science. Laplace as well as Weierstrass transforms are intimately related to the several indispensable concepts in diverse areas. The convolution of the transform plays an important role in digital signal processing. Seeing the importance of convolution structure of the transform in the development of science. The material I am presenting in this paper is the convolution structure of two dimensional Laplace-Weierstrass transform. Convolution theorem for Laplace-Weierstrass transform is proved also some properties of convolution for Laplace-Weierstrass transform is given.

Keywords*: Convolution theorem, Laplace-Weierstrass transform*

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1. INTRODUCTION:

The integral transform plays an important role in diverse areas. Fourier analysis is one of the most frequently used tool in signal processing and many other scientific disciplines. The Laplace as well as Weierstrass transform has been widely used in mathematical physics and applied mathematics.

The Laplace-Weierstrass transform is given by

$$LW\{f(t,y)\} = \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty e^{-st-\frac{(x-y)^2}{4}} f(t,y)\,dt\,dy$$

In our previous work Laplace-Weierstrass transform is defined in generalized sense. Also we have proved inversion theorem, uniqueness theorem, characterization theorem, representation theorem, analyticity theorem, disambiguation theorem.

Convolution is a mathematical operation defined on product of two functions. Convolution in one domain (e.g. time domain) equals point-wise multiplication in the other domain (e.g. frequency domain). Convolution theorem have many applications like atomic scattering factors, B-factors, diffraction from a lattice, diffraction from a crystal, resolution truncation, missing data, density modification also in diverse areas that includes probability, statistics, Computer vision, language processing, image processing, signal processing, engineering and many others.

Almeida [1] derived Fractional Fourier transforms convolution. Khairnar *et al.* [2] had described Bilateral Laplace-Mellin integral transform and its applications. In our previous work we proved inversion theorem and generalization of Laplace-Weierstrass transform [3,4]. Pathak [5] explained Integral transformation of generalized functions and their applications. Zayed [6] had revised the definition in order to follow the standard convolution theorem.

Convolution theorem for Laplace transform is defined as

$$(f*g)(t)=\int_0^t f(\Im)g(t-\Im)d\Im,\ t\geq 0 \tag{1.1}$$

Convolution theorem for Weierstrass transform is defined as

$$f(x)=(4\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-(x-y)^2/4}\phi(y)dy \tag{1.2}$$

This paper emphasizes on defining convolution for Laplace-Weierstrass transform. we established the convolution theorem in section [III]. We have given the some properties of convolution in section [IV]. Lastly conclusions are given in section [V].

Notation and terminologies were used as in Pathak [5] and Zemanian [7].

2. Definition Laplace-Weierstrass type convolution:

The convolution of two elements f and g belonging to $LW_{a,b}^{*}$ is defined by

$$\langle f*g,\phi\rangle=\langle f(t,y),\langle g(\xi,\eta),\phi(t+\xi,\ y+\eta)\rangle\rangle \tag{2.1}$$

For each $\phi\in LW_{a,b}$. This definition is meaningful provided that

$$\psi(t,y)=\langle g(\xi,\eta),\phi(t+\xi,\ y+\eta)\rangle \tag{2.2}$$

belongs to $LW_{a,b}$ and $f*g\in LW_{a,b}^{*}$. That this is, in fact, the case is shown in the next two theorems.

3. 1 Theorem 1:

Let $g(t,y)\in LW_{a,b}^{*}$, $\phi(t,y)\in LW_{a,b}$ and $\psi(t,y)$ be defined by (2.2). Then, $\psi(t,y)$ is infinitely differentiable and $D^{p}\,D^{q}\psi(t,y)=\langle g(\xi,\eta),\,D_t^{p}\,D_y^{q}\,\phi(t+\xi,y+\eta)\rangle$
$(3.1.1)$

Where $p,q=1,2,3$ - - - - - - -

Moreover $\psi(t,y)$ belongs to $LW_{a,b}$.

Proof:- We prove $D^{p}\,D^{q}\psi(t,y)=\langle g(\xi,\eta),\,D_t^{p}\,D_y^{q}\,\phi(t+\xi,y+\eta)\rangle$ for $p,q=1$. The general result follows by induction. For fixed $(t,y)\in R^2$ and $\Delta t,\Delta y\neq 0$, we have

$$\frac{1}{\Delta t\,\Delta y}[\psi(t+\Delta t,y+\Delta y)-\psi(t,y)]-\langle g(\xi,\eta),D_t\,D_y\phi(t+\xi,y+\eta)\rangle=\langle g(\xi,\eta),\theta_{\Delta t,\Delta y}(\xi,\eta)\rangle$$

Where $\theta_{\Delta t,\Delta y}(\xi,\eta)=\frac{1}{\Delta t\,\Delta y}[\phi(t+\Delta t+\xi,\ y+\Delta y+\eta)-\phi(t+\xi,y+\eta)]-D_tD_y\phi(t+\xi,\ y+\eta)$

To prove differentiability of $\psi(t,y)$ we have to show that $\theta_{\Delta t,\Delta y}(t,y)\to 0$ in $LW_{a,b}$ as $\Delta t,\Delta y\to 0$.

Using Taylor's formula with remainder we can write

$$\phi^{(p,q)}(t+\xi+\Delta t, y+\eta+\Delta y)=\phi^{(p,q)}(t+\xi, y+\eta)+\left[\Delta t\frac{\partial}{\partial t}\phi^{(p,q)}(t+\xi, y+\eta)+\Delta y\frac{\partial}{\partial y}\phi^{(p,q)}(t+\xi, y+\eta)\right]$$

$$+\frac{1}{2!}\left[\Delta t\frac{\partial}{\partial t}+\Delta y\frac{\partial}{\partial y}\right]^2\phi^{(p,q)}(t+\xi, y+\eta)+-----------$$

where $p,q=0,1,2----$

$$=\phi^{(p,q)}(t+\xi, y+\eta)+\left[\Delta t\,\phi^{(p+1,q)}(t+\xi, y+\eta)+\Delta y\,\phi^{(p,q+1)}(t+\xi, y+\eta)\right]$$

$$+\left[\begin{array}{l}\int_0^{\Delta t}(\Delta t-u)\phi^{(p+2,q)}(t+\xi+u, y+\eta)du\\ +4\int_0^{\sqrt{\Delta t}}\int_0^{\sqrt{\Delta y}}\left(\sqrt{\Delta t}-u\right)\left(\sqrt{\Delta y}-v\right)y\phi^{(p+1,q+1)}(t+\xi+u,\ y+\eta+v)du\,dv\\ +\int_0^{\Delta y}(\Delta y-v)\phi^{(p,q+2)}(t+\xi, y+\eta+v)dv\end{array}\right]$$

Therefore

$$\theta^{(p,q)}_{\Delta t,\Delta y}(\xi,\eta)=\frac{1}{\Delta t}[\int_0^{\Delta t}(\Delta t-u)\phi^{(p+2,q)}(t+\xi+u, y+\eta)du]+$$

$$\frac{4}{\sqrt{\Delta t}\sqrt{\Delta y}}[\int_0^{\sqrt{\Delta t}}\int_0^{\sqrt{\Delta y}}\left(\sqrt{\Delta t}-u\right)\left(\sqrt{\Delta y}-v\right)\phi^{(p+1,q+1)}(t+\xi+u,\ y+\eta+v)du\,dv]$$

$$+\frac{1}{\Delta y}[\int_0^{\Delta y}(\Delta y-v)\phi^{(p,q+2)}(t+\xi, y+\eta+v)dv]$$

Since for fixed t, y and all ξ,η and for $|\Delta t|<1$ and $|\Delta y|<1$

$$e^{a\xi-\frac{b\eta}{2}+\frac{\eta^2}{4}}\sup_{|u|\le|\Delta t|}\left|\phi^{(p+2,q)}(t+\xi+u, y+\eta)\right|,\quad e^{a\xi-\frac{b\eta}{2}+\frac{\eta^2}{4}}\sup_{\substack{|u|\le|\Delta t|\\|v|\le|\Delta y|}}\left|\phi^{(p+1,q+1)}(t+\xi+u, y+\eta+v)\right|$$

$$and\quad e^{a\xi-\frac{b\eta}{2}+\frac{\eta^2}{4}}\sup_{|v|\le|\Delta y|}\left|\phi^{(p,q+2)}(t+\xi, y+\eta+v)\right|$$

are bounded by the constant $C_{a,p}, C_{a,b,p,q}, C_{b,q}$ respectively, we have

$$\left|e^{a\xi-\frac{b\eta}{2}+\frac{\eta^2}{4}}\theta^{(p,q)}_{\Delta t,\Delta y}(\xi,\eta)\right|\le\left\{\frac{C_{a,p}}{\Delta t}\left|\int_0^{\Delta t}|\Delta t-u|\,du\right|+\frac{4C_{a,b,p,q}}{\sqrt{\Delta t}\sqrt{\Delta y}}\left|\int_0^{\sqrt{\Delta t}}\int_0^{\sqrt{\Delta y}}\left|\sqrt{\Delta t}-u\right|\left|\sqrt{\Delta y}-v\right|du\,dv\right|+\frac{C_{b,q}}{\Delta y}\left|\int_0^{\Delta y}|\Delta y-v|\,dv\right|\right\}$$

$$\le C_{a,p}\left|\frac{\Delta t}{2}\right|+C_{a,b,p,q}\left|\sqrt{\Delta t}\right|\left|\sqrt{\Delta y}\right|+C_{b,q}\left|\frac{\Delta y}{2}\right|\to 0$$

as $\Delta t, \Delta y \to 0$. Thus $\theta_{\Delta t, \Delta y}$ converges in $LW_{a,b}$ to zero as $\Delta t, \Delta y \to 0$. This proves differentiability of $\psi(t, y)$. Next, using the boundedness property of generalized functions, we have

$$\left| e^{at - \frac{by}{2} + \frac{y^2}{4}} D_t^p \, D_y^q \, \psi(t, y) \right| \le C e^{at - \frac{by}{2} + \frac{y^2}{4}} \max_{\substack{0 \le m \le r \\ 0 \le n \le r'}} \sup_{\substack{0<\xi<\infty \\ 0<\eta<\infty}} \left| e^{a\xi - \frac{b\eta}{2} + \frac{\eta^2}{4}} D_t^p \, D_y^q D_t^m \, D_y^n \, \phi(t+\xi, y+\eta) \right|$$

$$\text{Thus } \gamma_{p,q}(\psi) \le C \max_{\substack{0 \le m \le r \\ 0 \le n \le r'}} \gamma_{m+p, n+q}(\phi) \tag{3.1.2}$$

for each $p, q = 0,1,2 - - - - - -$ consequently $\psi(t, y) \in LW_{a,b}$. From equation (3.1.2) it follows that if $\{\phi_{v_1, v_2}\}_{v_1, v_2 \in N}$ also converges to zero in $LW_{a,b}$ then $\{\phi_{v_1, v_2}\}_{v_1, v_2 \in N}$ also converges to zero in $LW_{a,b}$. Thus altogether we have proved following.

3.2 Convolution Theorem 2:

Let $LW\{f(t, y)\} = F(s, x)$ for $s, x \in \Omega_f$ and $LW\{g(t, y)\} = G(s, x)$ for $s, x \in \Omega_g$. Assume further that $\Omega_f \cap \Omega_g$ is non-empty. Then

$$LW\{f * g\} = LW\{f(t, y)\} \cdot LW\left\{ g(t, y) \cdot e^{\frac{-y\eta}{2}} \right\} e^{\frac{x^2}{4}}, \quad s, x \in \Omega_f \cap \Omega_g$$

Proof: - Since $f * g \in LW_{a,b}^*$ it's Laplace-Weierstrass transform can be defined and on using

$$\langle f * g, \phi \rangle = \langle f(t, y), \langle g(\xi, \eta), \phi(t+\xi, y+\eta) \rangle \rangle, \tag{3.2.1}$$

We have

$$LW\{f * g\} = \frac{1}{4\pi} \left\langle (f * g)(t, y), e^{-st - \frac{(x-y)^2}{4}} \right\rangle$$

$$= \frac{1}{4\pi} \left\langle f(t, y), \left\langle g(\xi, \eta), e^{-s(t+\xi) - \frac{[x-(y+\eta)]^2}{4}} \right\rangle \right\rangle$$

$$= \frac{1}{\sqrt{4\pi}} \left[\left\langle f(t, y), e^{-st - \frac{(x-y)^2}{4}} \right\rangle \frac{1}{\sqrt{4\pi}} \left\langle g(\xi, \eta), e^{-s\xi - \frac{(x-\eta)^2}{4} + \frac{x^2}{4} - \frac{y\eta}{2}} \right\rangle \right]$$

$$LW\{f * g\} = LW\{f(t, y)\} \cdot LW\left\{ g(t, y) \cdot e^{\frac{-y\eta}{2}} \right\} e^{\frac{x^2}{4}}, \quad s, x \in \Omega_f \cap \Omega_g$$

4. Properties of convolution:

Let $LW\{f(t,y)\}=F(s,x), LW\{g(t,y)\}=G(s,x)$ *and* $LW\{h(t,y)\}=H(s,x)$ for $s,x\in\Omega_f,\ s,x\in\Omega_g,\ s,x\in\Omega_h$. Let $\Omega_f\cap\Omega_g\cap\Omega_h$ be non-empty, then

1) Commutativity:- $LW\{f*g\}=LW\{g*f\}$

2) Associativity:- $LW\{f*(g*h)\}=LW\{(f*g)*h\}$

3) $LW\{D_t D_y(f*g)\}=LW\{(D_t D_y f)*g\}=LW\{f*(D_t D_y g)\}$

5. CONCLUSION:-

In the present paper we proved convolution theorem and properties of convolution theorem for Laplace-Weierstrass transform.

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