SIGNED AND ROMANCED DOMINATING FUNCTIONS OF
CORONA PRODUCT GRAPH OF CYCLE WITH A COMPLETE
GRAPH

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ABSTRACT

Graph theory is one of the most flourishing branches of modern mathematics and finds its applications to various branches of Science & Technology. Domination in graphs has been studied extensively in recent years and it is an important branch of graph theory. An introduction and an extensive overview on domination in graphs and related topics is surveyed and detailed in the two books by Haynes et al. [10,11]. In this paper we present some results on minimal signed edge dominating functions and Roman edge dominating functions of corona product graph of cycle with a complete graph.

Keywords: Corona Product, signed edge dominating function, Roman edge dominating function.

1. INTRODUCTION

Domination Theory has a wide range of applications to many fields like Engineering, Communication Networks, Social sciences, linguistics, physical sciences and many others. Allan, R.B. and Laskar, R. [1], Cockayne, E.J. and Hedetniemi, S.T. [5] have studied various domination parameters of graphs.

Frucht and Harary [9] introduced a new product on two graphs \(G_1\) and \(G_2\), called corona product denoted by \(G_1 \odot G_2\). The object is to construct a new and simple operation on two graphs \(G_1\) and \(G_2\) called their corona, with the property that the group of the new graph is in general isomorphic with the wreath product of the groups of \(G_1\) and of \(G_2\).

The concept of edge domination was introduced by Mitchell and Hedetniemi [15] and it is explored by many researchers. Arumugam and Velammal [4] have discussed the edge domination in graphs while the fractional edge domination in graphs is discussed in Arumugam and Jerry [3]. The complementary edge domination in graphs is studied by Kulli and Soner [14] while Jayaram [13] has studied the line dominating sets and obtained bounds for the line domination number. The bipartite graphs with equal edge domination number and maximum matching cardinality are characterized by Dutton and Klostermeyer [7] while Yannakakis and Gavril [17] have shown that edge dominating set problem is NP-complete even when restricted to planar or bipartite graphs of maximum degree. The edge domination in graphs of cubes is studied by Zelinka [18].

2. Corona product graph \(C_n \odot K_m\)

The corona product of a cycle \(C_n\) with a complete graph \(K_m\) is a graph obtained by taking one copy of a \(n\) – vertex graph \(C_n\) and \(n\) copies of \(K_m\) and then joining the \(i^{th}\) vertex of \(C_n\) to every vertex of \(i^{th}\) copy of \(K_m\). This graph is denoted by \(C_n \odot K_m\).

The vertices of \(C_n\) are denoted by \(v_1, v_2, \ldots, v_n\). The edges in \(C_n\) are denoted by \(e_1, e_2, \ldots, e_n\) where \(e_i\) is the edge joining the vertices \(v_i\) and \(v_{i+1}\), \(i \neq n\). For \(i = n, e_n\) is the edge joining the vertices \(v_n\) and \(v_1\).
The vertices in the \( i^{th} \) copy of \( K_m \) are denoted by \( w_{ij}, i = 1, 2, ..., \frac{m(m-1)}{2} \).

The edges in the outer Hamilton cycle of the \( i^{th} \) copy of \( K_m \) are labelled by \( l_{ij} \) where \( l_{ij} \) is the edge joining the vertices \( w_{ij} \) and \( w_{i(j+1)}, i = 1, 2, ..., (m - 1) \). For \( j = m, l_{im} \) is the edge joining the vertices \( w_{im} \) and \( w_{11} \). There are another type of edges of \( \mathcal{C}_n \) which are denoted by \( h_{ij}, h_{i1}, h_{i2}, ..., h_{im} \) and these are adjacent to each other and incident with the vertex \( v_i \) of \( \mathcal{C}_n \).

The basic properties and the edge dominating sets of \( G = \mathcal{C}_n \odot K_m \) are presented in Anitha [2].

**Theorem 2.1:** The adjacency of an edge \( e \) in \( G = \mathcal{C}_n \odot K_m \) is given by

\[
\text{adj}(e) = \begin{cases} 
2m + 2, & \text{if } e = e_i \in \mathcal{C}_n, \\
2m - 2, & \text{if } e = l_{ij} \in i^{th} \text{ copy of } K_m, \\
2m, & \text{if } e = h_{ij} \in G = \mathcal{C}_n \odot K_m.
\end{cases}
\]

**3. Signed Edge Dominating Functions**

The concept of Signed dominating function was introduced by Dunbar et al., [8]. There is a variety of possible applications for this variation of domination. By assigning the values \(-1\) or \(+1\) to the vertices of a graph we can model such things as networks of positive and negative electrical charges, networks of positive and negative spins of electrons and networks of people or organizations in which global decisions can be made.

In this section, some results on minimal signed edge dominating functions of the graph \( G = \mathcal{C}_n \odot K_m \) is obtained. Let us recall the definitions of signed edge dominating function and minimal signed edge dominating function of a graph \( G(V, E) \).

**Definition:** Let \( G \ (V, E) \) be a graph. A function \( f : E \rightarrow \{-1, 1\} \) is called a signed edge dominating function (SEDf) of \( G \) if

\[
f(N[e]) = \sum_{e \in E(G)} f(e) \geq 1, \forall e \in E(G).
\]

A signed edge dominating function \( f \) of \( G \) is called a minimal signed edge dominating function (MSEDF) if for all \( g < f, g \) is not a signed edge dominating function.

**Theorem 3.1:** A function \( f : E \rightarrow \{-1, 1\} \) defined by

\[
f(e) = \begin{cases} 
-1, & \text{for } (m - 1) \text{ edges } e = l_{ij} \text{ in each copy of } K_m, \\
1, & \text{otherwise.}
\end{cases}
\]

is a minimal signed edge dominating function of \( G = \mathcal{C}_n \odot K_m \).

**Proof:** Let \( f \) be a function defined as in the hypothesis. By the definition of the function \(-1\) is assigned to \((m - 1)\) edges \( l_{ij} \) in each copy of \( K_m \) in \( G \) and \( 1 \) is signed to remaining edges of \( G \).

The summation value taken over \( N[e] \) of \( e \in E \) is as follows.

**Case 1:** Let \( e_i \in \mathcal{C}_n \), be such that \( \text{adj}(e_i) = 2m + 2 \) in \( G \). Then \( N[e_i] \) contains three edges of \( \mathcal{C}_n \) and \( 2m \) edges which are drawn from the vertices \( v_i \) and \( v_i+1 \) respectively to the \( m \) vertices of \( i^{th} \) and \((i + 1)^{th} \) copies of \( K_m \) and their functional value is \( 1 \).

Therefore

\[
\sum_{e \in N[e_i]} f(e) = 1 + 1 + 1 + \underbrace{1 + 1 + \cdot \cdot \cdot + 1}_{2m\text{-times}} = 2m + 3.
\]

**Case 2:** Let \( l_{ij} \in i^{th} \text{ copy of } K_m \). By the definition of \( f, (m - 1)\) edges of \( l_{ij} \) are assigned \(-1\) and the remaining edges are assigned \( 1 \). By Theorem 2.1, \( \text{adj}(l_{ij}) = 2m - 2 \). That is the edge \( l_{ik} \) is adjacent to \((2m - 4)\) edges \( l_{ij}, j \neq k \) and two edges \( h_{ij} \). Here \( f(h_{ij}) = 1 \).
So \[ \sum_{e \in N[l_{ij}]} f(e) = [(m - 1)(-1) + (m - 2)(1)] + 1 + 1 = 1. \]

Now for all other possibilities of functional values of \( l_{ij} \) that are adjacent to \( t_{ik}, \) we could see that \( \sum_{e \in N[l_{ij}]} f(e) > 1. \)

**Case 3:** Let \( h_{ik} \in C_n \cap K_m \) be such that \( \text{adj}(h_{ij}) = 2m \) in \( G. \) Then \( N[h_{ij}] \) contains two edges of \( C_n, \) \( m \) edges \( h_{ij} \) and \((m - 1)\) edges \( l_{ij} \) in \( K_m. \)

Suppose \( f(l_{ij}) = -1 \) for all \((m - 1)\) edges \( l_{ij} \) that are adjacent to \( h_{ij}. \) Then
\[
\sum_{e \in N[h_{ij}]} f(e) = [(m - 1)(-1) + (m)(1)] + 1 + 1 = 3.
\]

Suppose \( f(l_{ij}) = 1 \) for all \((m - 1)\) edges \( l_{ij} \) that are adjacent to \( h_{ij}. \) Then
\[
\sum_{e \in N[h_{ij}]} f(e) = [(m - 1)(1) + (m)(1)] + 1 + 1 = 2m + 1.
\]

Thus as in Case 2 for all other possibilities of functional values for the \((m - 1)\) edges that are adjacent to \( h, \) we could see that
\[
\sum_{e \in N[h_{ij}]} f(e) > 1.
\]

Therefore for all possibilities we get
\[
\sum_{e \in E(G)} f(e) \geq 1, \text{ for all } e \in E(G).
\]

Hence \( f \) is an edge dominating function.

We now check for the minimality of \( f. \)

Define a function \( g : E \rightarrow \{-1, 1\} \) by
\[
g(e) = \begin{cases} 
-1, & \text{for one edge } h_{ik}, \\
-1, & \text{for } (m - 1)\text{ edges } l_{ij} \text{ in each copy of } K_m, \\
1, & \text{otherwise}.
\end{cases}
\]

Since strict inequality holds at \( h_{ik} \) it follows that \( g < f. \)

**Case (i):** Let \( e_i \in C_n \) be such that \( \text{adj}(e_i) = 2m + 2 \) in \( G. \)

**Sub Case 1:** Let \( h_{ik} \in N[e_i]. \) Then
\[
\sum_{e \in N[e_i]} g(e) = -1 + 1 + 1 + \frac{[1 + 1 + \cdots + 1]}{2m-\text{times}} = 2m + 1.
\]

**Sub Case 2:** Let \( h_{ik} \notin N[e_i]. \) Then
\[
\sum_{e \in N[e_i]} g(e) = 1 + 1 + 1 + \frac{[1 + 1 + \cdots + 1]}{2m-\text{times}} = 2m + 3.
\]

**Case (ii):** Let \( l_{ij} \in t^{\text{th}} \text{ copy of } K_m. \) Then \( \text{adj}(l_{ij}) = 2m - 2 \) in \( G. \)

**Sub Case 1:** Let \( h_{ik} \in N[l_{ij}]. \) Then
\[
\sum_{e \in N[l_{ij}]} g(e) = [(m - 1)(-1) + (m - 1)(1)] + (-1) = -1.
\]

**Sub Case 2:** Let \( h_{ik} \notin N[l_{ij}]. \) Then
\[
\sum_{e \in N[l_{ij}]} g(e) = [(m - 1)(-1) + (m - 1)(1)] + 1 = 1.
\]
Now for all other possibilities of functional values of $l_{ij}$ that are adjacent to $l_{ik}$ we could see that
\[ \sum_{e \in N[l_{ik}]} g(e) < 1. \]

**Case(iii):** Let $h_{ij} \in C_n \otimes K_m$ be such that $adj(h_{ij}) = 2m$ in $G$. The edge $h_{ij}$ is adjacent to
(m − 1) edges $l_{ij}$ in each copy of $K_m$. Then as in Case 3, for all possibilities of functional values for the (m − 1) edges that are adjacent to $h_{ij}$, we can see that
\[ \sum_{e \in N[h_{ij}]} g(e) > 1. \]
But, as in Case(ii) we have
\[ \sum_{e \in E[G]} g(e) < 1, \text{ for some } e \in E. \]
So $g$ is not an edge dominating function. Since $g$ is defined arbitrarily, it follows that there exists no $g < f$ such that $g$ is an edge dominating function. Thus $f$ is a minimal signed edge dominating function. ■

**4. Roman Edge Dominating Function**

The Roman dominating function of a graph $G$ was defined by Cockayne et al. [6]. The definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart [12] entitled “Defend the Roman Empire!” and suggested by even earlier by ReVelle[16].

In this section, first we recall the definitions of Roman dominating function of a graph. Later results on minimal Roman edge dominating functions of $G = C_n \otimes K_m$ are obtained.

**Definition:** Let $G (V, E)$ be a graph. A function $f : E \to \{0, 1, 2\}$ is called a **Roman edge dominating function (REDF)** of $G$ if
\[ f(N[e]) = \sum_{e \in E(G)} f(e) \geq 1, \forall e \in E(G) \]
and satisfying the condition that every edge $e$, for which $f(e) = 0$ is adjacent to at least one edge $e'$ for which $f(e') = 2$.

A Roman edge dominating function $f$ of $G$ is called a **minimal Roman edge dominating function (MREDF)** if for all $g < f$, $g$ is not a Roman edge dominating function.

**Theorem 4.1:** A function $f : E \to \{0, 1, 2\}$ defined by
\[ f(e) = \begin{cases} 2, & \text{for } e = h_{ij} \text{ in } C_n \otimes K_m \text{ for all } i \text{ and } j, \\ 0, & \text{otherwise.} \end{cases} \]
is a minimal Roman edge dominating function of $G = C_n \otimes K_m$.

**Proof:** Let $f$ be a function defined as in the hypothesis.

**Case 1:** Let $e_i \in C_n$ be such that $adj(e_i) = 2m + 2$ in $G$. Then $N[e_i]$ contains $m$ edges $h_{i1}, h_{i2}, ..., h_{im}$ of $G$. $m$ edges $h_{(i+1)1}, h_{(i+1)2}, ..., h_{(i+1)m}$ of $G$ and three edges of $C_n$.

So \[ \sum_{e \in N[e_i]} f(e) = 0 + 0 + 0 + \left[ 2 + 2 + \cdots + 2 \right] = 4m. \]

**Case 2:** Let $l_{ij} \in i^{th}$ copy of $K_m$. Then $adj(l_{ij}) = 2m - 2$ in $G$. Then $N[l_{ij}]$ contains $(2m - 3)$ edges of $K_m$ and two edges $h_{ij}$ of $G$. 

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So \( \sum_{e \in N[l_{ij}]} f(e) = 2 + 2 + \left[ 0 + 0 + \cdots + 0 \right] = 4. \)

**Case 3:** Let \( h_{ij} \in C_n \odot K_m \) be such that \( \text{adj}(h_{ij}) = 2m \) in \( G \).

So \( \sum_{e \in N[h_{ij}]} f(e) = 0 + 0 + [(m - 1)0 + (m \times 2)] \)

\( = 2m. \)

Therefore, for all possibilities, we get

\[ \sum_{e \in E[G]} f(e) > 1. \]

Let \( e \) be an edge of \( G \) such that \( f(e) = 0 \) and \( e' \) be another edge of \( G \) such that \( e' \neq e \) and \( f(e') = 2 \). Then we show that \( e \) and \( e' \) are adjacent.

Now \( f(e) = 0 \) implies \( e = e_i \in C_n \) for some \( i \), or \( e = l_{ij} \), for some \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \).

Now \( f(e') = 2 \) implies \( e' = h_{ij}, \ i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m. \)

Suppose \( e = e_i \in C_n. \) Then obviously \( e_i \) and \( h_{ij} \) are adjacent, that is \( e \) and \( e' \) are adjacent.

Suppose \( e = l_{ij}, \) for some \( i \) and \( j. \) Then \( l_{ij} \) and \( h_{ij} \) are adjacent. That is \( e \) and \( e' \) are adjacent.

This implies that \( f \) is a Roman edge dominating function.

Now we check for the minimality of \( f. \) Define a function \( g : E \to \{0, 1, 2\} \) by \( g(e) = \begin{cases} 
1, \text{for two edges } h_{ik}, h_{ij} \in C_n \odot K_m, & \text{for all edges } h_{ij} \in C_n \odot K_m, j \neq k, l \\
0, \text{otherwise.} & \end{cases} \)

Since strict inequality holds at an edge \( h_{ik}, \) it follows that \( g < f. \)

**Case (i):** Let \( e_i \in C_n \) be such that \( \text{adj}(e_i) = 2m + 2 \) in \( G. \)

**Sub Case 1:** Let \( h_{ik} \in N[e_i]. \) Then

\[ \sum_{e \in N[e_i]} g(e) = 0 + 0 + 0 + \left[ 2 + 2 + \cdots + 2 \right] + 1 + 1 = 4m - 2. \]

**Sub Case 2:** Let \( h_{ik} \notin N[e_i]. \) Then

\[ \sum_{e \in N[e_i]} g(e) = 0 + 0 + 0 + \left[ 2 + 2 + \cdots + 2 \right] = 4m. \]

**Case (ii):** Let \( l_{ij} \in i^{th} \) copy of \( K_m. \) Then \( \text{adj}(l_{ij}) = 2m - 2 \) in \( G. \)

**Sub Case 1:** Let \( h_{ik} \in N[l_{ij}]. \) Then

\[ \sum_{e \in N[l_{ij}]} g(e) = \left[ 0 + 0 + \cdots + 0 \right] + 1 + 1 = 2, \]

or

\[ \sum_{e \in N[l_{ij}]} g(e) = \left[ 0 + 0 + \cdots + 0 \right] + 2 + 1 = 3. \]

**Sub Case 2:** Let \( h_{ik} \notin N[l_{ij}]. \) Then \( m \neq 3. \)

\[ \sum_{e \in N[l_{ij}]} g(e) = \left[ 0 + 0 + \cdots + 0 \right] + 2 + 2 = 4. \]

or

\[ \sum_{e \in N[l_{ij}]} g(e) = \left[ 0 + 0 + \cdots + 0 \right] + 2 + 1 = 3. \]
When \( m = 3 \), the possibility \( h_{ik} \notin N[l_{ij}] \) does not arise.

**Case (iii):** Let \( h_{ij} \in C_n \odot K_m \) be such that \( \text{adj}(h_{ij}) = 2m \) in \( G \).

Let \( h_{ik} \in N[h_{ij}] \). Then

\[
\sum_{e \in N[h_{ij}]} g(e) = 0 + 0 + \left[ 2 + 2 + \cdots + 2 \right] + 1 + 1 = 4m - 4.
\]

Similar is the case for the edge \( h_{il} \).

Hence for all possibilities, we get

\[
\sum_{e \in E(G)} g(e) > 1, \text{ for all } e \in E(G).
\]

i.e. \( g \) is an edge dominating function. But \( g \) is not a Roman edge dominating function, since the REDF definition fails in the \( i^{th} \) copy of \( K_m \) in \( G \).

Let the edge \( l_{ij} \in i^{th} \) copy of \( K_m \). Then \( g(l_{ij}) = 0 \).

We know that every edge \( l_{ij} \) in \( K_m \) is adjacent to two edges \( h_{ij}, f = 1, 2, \ldots, m. \)

The condition of Roman dominating function fails for the edge \( l_{ij} \) which is adjacent to \( h_{ik} \) and \( h_{il} \), as \( g(h_{ik}) = g(h_{il}) = 1 \).

Thus \( f \) is a minimal Roman edge dominating function.

**5. Illustrations:**

**MINIMAL SIGNED EDGE DOMINATING FUNCTION**

**Theorem 3.1**

The functional values are given at each edge of the graph \( G \).

**MINIMAL ROMAN EDGE DOMINATING FUNCTION**

**Theorem 4.1**

The functional values are given at each edge of the graph \( G \).
6. REFERENCES
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